Roshko[4] for vortex shedding in steady flow. Our data points in general are lower than the Roshko curve, however, they do show a definite correlation between the instantaneous Strouhal and Reynolds numbers. These results suggest the following conclusions: 1) In an oscillatory freestream of 3 Hz and Reynolds number up to 4×10^4, the vortex shedding from a circular cylinder responds instantaneously to the freestream variations. 2) In the instantaneous Reynolds number range of 500 to 4×10^4, the instantaneous Strouhal number stays sensibly constant at 0.20, ± 0.01. 3) With a limited set of data points in the instantaneous Reynolds number range of 3 to 8×10^3, the results show no systematic variation when the frequency is increased from 3 to 6 Hz.

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Comparison of Theory and Experiment for Ion Collection by Spherical and Cylindrical Probes in a Collisional Plasma

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This Note describes double probe ion saturation current measurements made with spherical and cylindrical probes in a helium negative glow plasma operated in the transitional pressure range for the probe sizes tested. The results are compared with two currently available transitional regime probe theories. In addition a relatively simple formulation is described which approximates the more rigorous theories.

If a formal sheath boundary is postulated, the current I_p to a negatively biased probe in a plasma that contains a single positive ion species can be expressed in terms of the sheath ion flux; i.e.,

$$I_p = e\phi A_s n_s u_s = e\phi A_p n_0 u_s (A_s/A_p)(n_s/n_0) \tag{1}$$

where n is the density and u the velocity of the ions, A the surface area, e the electronic charge, and ϕ the fraction of ions crossing the sheath that reach the probe. The subscripts s and p refer to the sheath and probe surfaces. Since ion current measurements are generally used to estimate the plasma electron (ion) density n_0, suitable theories must relate $u_s\phi(A_s/A_p)(n_s/n_0)$ to parameters, such as the electron and ion temperatures (T_e and T_i) and the gas density, which can be determined by other techniques.

Laframboise[1] treated the collisionless case for spherical and cylindrical probes. His results—presented as a tabulation of a parameter i such that

$$I_p = eA_p n_0 (kT_e/2\pi M_i)^{1/2} i(V_p/kT_e, r_p/\lambda_d, T_i/T_e) \tag{2}$$

where V_p is the probe potential (relative to the plasma), r_p the probe radius, and λ_d the Debye length—have been verified experimentally, for the cylindrical case, by comparison with microwave measurements.[2] Kiel[3] has reviewed the collision dominated case ($\lambda_i \ll \lambda_d$, where λ_i is ion mean free path) and derived approximated relations which can be written as

$$(I_p)_{\text{sphere}} = eA_p n_0 (A_s/A_p)_c \, \mu_i k(T_e + T_i)/r_s \tag{3}$$

$$(I_p)_{\text{cyl}} = eA_p n_0 (A_s/A_p)_c \, [\mu_i k(T_e + T_i)/r_s] \, 1/\ln(\pi L/4r_s) \tag{4}$$

where μ_i is the ion mobility, L is the length of a cylindrical probe, and semiempirical expressions are given for r_s/r_p. Several attempts have been made to develop a sufficiently general theory to handle the transition regime between the collisionless and collision dominated cases.[4–7] The theory of Chou et al.,[4] although difficult to use, is believed to be the most rigorous. From this theory Talbot and Chou[5] have constructed manageable approximations for both spherical and cylindrical probes. Self and Shih[6] have presented useful data for spherical probes. Although the Talbot-Chou and Self-Shih theories are in general agreement, yielding the Laframboise results in the collisionless limit, and essentially the result approximated by Kiel in the collision dominated limit, the Talbot-Chou theory predicts slightly lower currents in the transitional range (see Figs. 1 and 2). Waymouth[7] formulated a theory for the spherical probe thin sheath case by solving the diffusion equations in the quasi-neutral region and matching the solution to the collisionless thermal flux at the sheath edge. His results do not agree with Laframboise's in the collisionless limit because of a failure to include ion inertia effects in the diffusion equations.

An examination of the published numerical results for the collisionless and collision dominated cases permits the following observations for negatively biased probes of practical size for laboratory plasmas ($r_p/\lambda_d > 10$, $\phi = 1$). 1) The probe current is controlled primarily by the electric field in the quasi-neutral region. In the collisionless case this field causes a moderate density decrease ($n_s/n_0 \sim 0.5$) and a significant ion acceleration [$u_s \sim (kT_e/M_i)^{1/2}$]. In the collision dominated case the quasi-neutral field extends farther into the plasma, the density decrease is large ($n_s/n_0 \sim 0.1$), and the ion acceleration should remain significant.[8,9] 2) The primary significance of the sheath edge per se is in determining the effective capture area of the probe. These observations suggest that a relatively simple transition regime probe theory can be formulated by matching a collision dominated quasi-neutral region with a collisionless inertia controlled formulation such as that of Laframboise.[1] This approach is possible because as collisions become important their first-order effect is to increase the density drop in the quasi-neutral region, thereby effectively establishing new boundary conditions for the collisionless sheath. When the density is so high that collisions become important in the sheath, the ion transport is dominated by the large density drop in the quasi-neutral region, and the model's assumption of a collisionless sheath does not introduce a large error. A similar situation holds for u_s. In the collision dominated limit u_s cancels out of the formulation. As one passes into the transition regime inertia effects become increasingly important in the quasi-neutral region, and the assumption that u_s is approximately equal to $(kT_e/2\pi M_i)^{1/2}$ (the value implied by the collisionless theory) appears realistic.[8,9]

Consider the case of a spherical probe. The collision induced density drop across the quasi-neutral region of a negatively biased probe with a collisionless sheath can be written as[7]

$$n_s/n_0 = 1/[1 + u_s r_s/\mu_i k(T_e + T_i)] \tag{5}$$

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Combining Eq. (5) with Eq. (1) yields for $\phi = 1$

$$I_p = eA_p n_0 u_s (A_s/A_p)\ 1/[1 + u_s\, r_s/\mu_i k(T_e + T_i)] \quad (6)$$

In the collisionless limit Eq. (6) reduces to $I_p = eA_p n_0 u_s(A_s/A_p)$, which is set equal to the collisionless formulation expressed by Eq. (2). This is equivalent to recognizing that u_s is approximately equal to $(kT_e/2\pi M_i)^{1/2}$, and that i expresses primarily the sheath area ratio. After this manipulation Eq. (6) differs from Eq. (3) in the collision dominated limit only by the factor $i(A_p/A_s)_c$—i.e., essentially the difference between the collisionless and collision dominated sheath area ratios. Although this variation is not large, improved accuracy can be obtained by extrapolating the collisionless to collision dominated sheath variation across the transitional regime. Accordingly, the final form of Eq. (6) is

$$I_p = eA_p n_0 \left(\frac{kT_e}{2\pi M_i}\right)^{1/2} \times$$
$$i\, \frac{1}{1 + (kT_e/2\pi M_i)^{1/2}(A_p/A_s)_c\, i\, r_s/\mu_i k(T_e + T_i)} \quad (7)$$

which can be written in the simple form

$$I_p = I_0/(1 + I_0/I_c) = I_0 I_c/(I_0 + I_c) \quad (8)$$

where I_0 and I_c are the collisionless and collision dominated currents to the probe in question, i.e., for the spherical probe the currents given by Eqs. (2) and (3). As shown by the form of Eq. (8), sheath matching yields the same results as would be obtained by imagining that the probe collection was controlled by two series electrical impedances, one corresponding to the collisional drag and the other corresponding to the ion inertia. Equation (8) permits one to approximate the transition regime current to a probe of any geometry for which expressions for the collisionless and collision dominated current are available. Thus substitution of Eqs. (2) and (4) provides a suitable expression for the cylindrical probe. The spherical and cylindrical cases are seen in Figs. 1 and 2 to be in excellent agreement with the results obtained in the more rigorous analysis.

Very limited experimental results are available for comparison with the transition regime theories. However, where comparisons have been possible the results are encouraging for cylindrical probes[2,10] and for spherical probes at $\lambda_i/r_p \geq 1$.[6] There is need for additional experimental data, particularly for spherical probes in the region of strong collisional effects.[3,10]

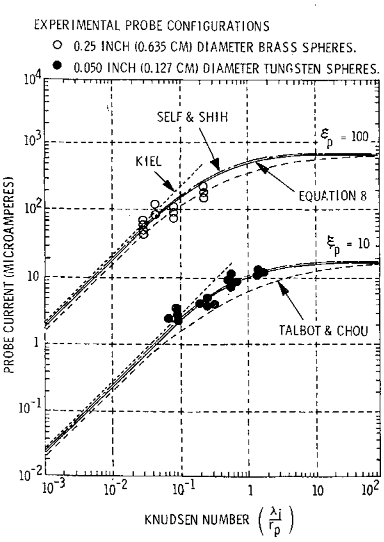

Fig. 1 Comparison of theory and experiment, $\xi_p = r_p/\lambda_d$, helium mobility of 8 × 10³ cm²/v-sec (1 torr and 0°C) used in calculations.

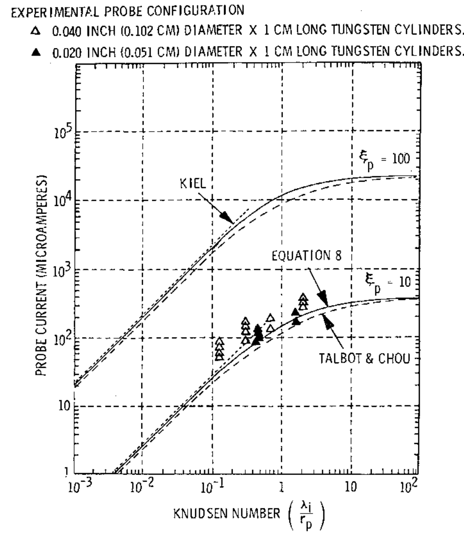

Fig. 2 Comparison of theory and experiment, $\xi_p = r_p/\lambda_d$, helium mobility of 8 × 10³ cm²/v-sec (1 torr and 0°C) used in calculations.

Experiments were performed in a helium negative glow generated by a 4-in.-diam aluminum cathode placed in one arm of a six sided Pyrex cross. Helium was flowed continuously through the system and pumped by a suitably trapped oil diffusion pump. The pressure was monitored by thermocouple gages calibrated against an MKS capacitive manometer. Two arms of the cross were used for an X-band microwave interferometer. Double probes, of sizes selected to minimize the range of plasma conditions required for obtaining the desired range of Knudsen numbers (spherical probes of 0.050, 0.100, and 0.250 in. diam; 1-cm-long cylindrical probes of 0.005, 0.020 and 0.040 in. diam), were inserted in the remaining arms and arranged so that they could alternatively be positioned in a central volume of the plasma which was found to have a nearly uniform electron density. Probe characteristics were generated manually and recorded on an X-Y recorder. An auxiliary circuit permitted the probes to be cleaned at any point in their characteristic by biasing them relative to the anode and far into the electron saturation region.

Experiments were conducted at conditions (pressure, 50 to 700 μ; discharge current, 50 to 500 ma) which provided electron densities in the range 5×10^8–10^{11} cm^{-3}.† Since the microwave interferometer permitted electron density measurements over the approximate range 5×10^{10} cm^{-3} to 10^{12} cm^{-3}, comparisons between the probe and microwave measurements were limited to the higher electron densities. Accordingly the microwave measurements were accompanied by single probe measurements of the electron temperature and density, using a probe of sufficiently small size (0.005 in. diam) so that it remained in the collisionless regime for most of the experiments. The collisionless single probe measurements were verified (to within 20%) at the higher electron densities by the microwave measurements. Fortunately at high pressures, where the single probes were apparently subject to collisional effects, the electron densities were of sufficient magnitude to permit microwave measurements.

Measured ion saturation currents for r_p/λ_d of 10 and 100 are shown in Figs. 1 and 2. The spherical probe data, which are in the Knudsen number range where little experimentation has been reported, are in general agreement with the theories described above. In order to permit comparison on a single curve for $r_p/L = 100$, the cylindrical probe data were multi-

† Although the negative glow was maintained by energetic primary electrons emanating from the cathode, the volume of the plasma was composed almost entirely of relatively lower energy (0.3 ev) ultimate electrons.

plied by 100 L/r_p. The resulting currents exhibit the expected dependence on the Knudsen number but are large by a factor of about two. It is believed that this is an end effect, since the end-on cross-sectional area of the sheath for the experimental probes was of the same order as the circumferential area.

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Implicit Rigid Body Motion in Curved Finite Elements

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It is generally conceded that rigid body motion must be adequately represented in curved as well as flat finite elements. However, there is still much disagreement regarding whether the rigid body motion should be represented explicitly or whether an implicit representation is sufficient. This Note pertains to the implicit representation where the rigid body modes are resurrected with mesh refinement.

Stricklin, Navaratna and Pian[1] ignored the explicit representation of rigid body motion in the development of a curved element for a shell of revolution. The authors used a cubic function in the meridional distance for the normal displacement and linear functions for the displacements in the meridional and circumferential directions. They simply stated that the explicit inclusion of rigid body motion was not necessary. It was later shown by Haisler and Stricklin[2] that rigid body motion is, in fact, recovered with mesh refinement. In certain cases the recovery may however be too slow to be of practical value. Gallagher[3] and Schmit, Bogner, and Fox[4] later developed curved cylindrical elements for the analysis of shells by the direct stiffness method. In Ref. 4 it was found that linear displacement functions were insufficient for the adequate representation of rigid body motion and consequently, bicubic functions were used for all displacements. More recently many authors have developed finite elements based on the implicit inclusion of rigid body motion. References 5 and 6 present reviews of these elements.

The curved shell of revolution element presented in Ref. 1 has been used in computer codes developed at Texas A&M University[7,8] and at M.I.T.[9,10] Experience has shown that rigid body motion is represented with element refinement. However, recent experience with shells which circumscribe very large angles (>90°) has demonstrated that a large number of elements (>50) may be required for an adequate solution. This offers no difficulty for linear analyses but requires the expenditure of large amounts of computer time for nonlinear analyses. Consequently, a better element was developed and is reported in the present Note.

The curvature of the shell is represented by the same procedure as given in Ref. 1. However, in the present research the displacements are given by

$$
\begin{aligned}
w &= \Sigma(\alpha_1{}^i + \alpha_2{}^i S + \alpha_3{}^i S^2 + \alpha_4{}^i S^3)\cos(i\theta) \\
u &= \Sigma[\alpha_5{}^i + \alpha_6{}^i S + \beta_1{}^i S(S-L) + \\
&\qquad \beta_2{}^i S^2(S-L)]\cos(i\theta) \qquad (1) \\
v &= \Sigma[\alpha_7{}^i + \alpha_8{}^i S + \beta_3{}^i S(S-L) + \\
&\qquad \beta_4{}^i S^2(S-L)]\sin(i\theta)
\end{aligned}
$$

where w, u, and v are the displacements in the normal, meridional, and circumferential directions, respectively, and L is the length of the element in the meridional direction. S is the meridional distance along the element.

Substituting Eqs. (1) into the strain energy expression, the internal energy may be written as

$$
U = \frac{1}{2}\,[\alpha \quad \beta]\begin{bmatrix} L_{\alpha\alpha} & L_{\alpha\beta} \\ L_{\beta\alpha} & L_{\beta\beta} \end{bmatrix}\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \qquad (2)
$$

Since the coefficients of the β terms vanish at both ends of the elements, the β terms may be eliminated by static condensation. This yields an element stiffness matrix in terms of the α coefficients.

$$
[L] = [L_{\alpha\alpha}] - [L_{\alpha\beta}][L_{\beta\beta}]^{-1}[L_{\beta\alpha}] \qquad (3)
$$

The transformation to global directions follows the same procedure as given in Ref. 7.

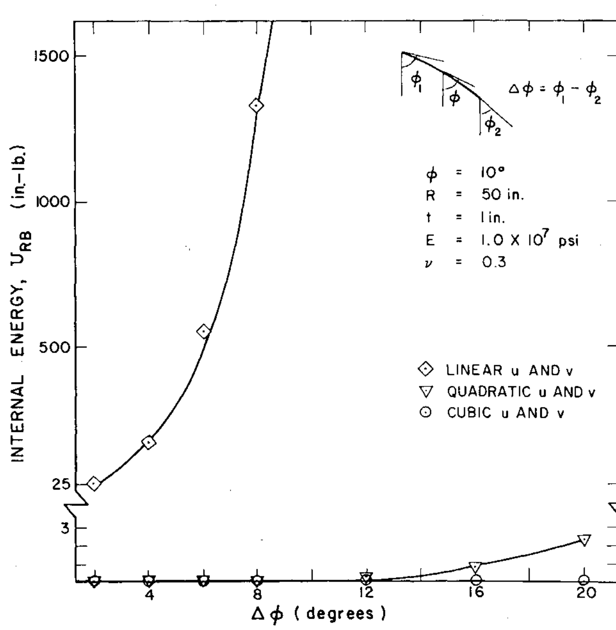

Fig. 1 Internal energy associated with an axial rigid body motion of a hemispheric element located near the base.

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